# *Change of the Independent Variable.*

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BY Taylor's Theorem, if $u \equiv f(y)$ and $x \equiv \phi(y)$,

$$f(y+h) = u + hd_y u + h^2 \frac{d_y^2 u}{2\,!} + h^3 \frac{d_y^3 u}{3\,!} + \text{etc.}; \qquad \text{(A)}$$

also $$f(y+h) = f\phi^{-1}[x + \{\phi(y+h) - x\}]$$

$$= u + \{\phi(y+h)-x\} d_x u + \{\phi(y+h)-x\}^2 \frac{d_x^2 u}{2\,!} + \{\phi(y+h)-x\}^3 \frac{d_x^3 u}{3\,!} + \text{etc.} \qquad \text{(B)}$$

$$\phi(y+h) = x + hd_y x + h^2 \frac{d_y^2 x}{2\,!} + h^3 \frac{d_y^3 x}{3\,!} + \text{etc.} \qquad \text{(C)}$$

Substitute by (C) for $\{\phi(y+h) - x\}$ in (B), and equate coefficients of like powers of $h$ in (A) and (B) thus reduced.

For convenience, let

$$x_1 \equiv d_y x, \qquad x_2 \equiv \frac{d_y^2 x}{2\,!}, \qquad x_3 \equiv \frac{d_y^3 x}{3\,!}, \dots$$

$$u_1 \equiv d_y u, \qquad u_2 \equiv \frac{d_y^2 u}{2\,!}, \qquad u_3 \equiv \frac{d_y^3 u}{3\,!}, \dots$$

and $S_m^n \equiv$ the sum of the terms of weight $m$ in the expansion of

$$(0_0 + x_1 + x_2 + x_3 + \dots)^n$$

(from which it immediately follows that, if $p > m$, $S_m^p = 0$ and that $S_m^m = x_1^m$).

$$u_1 = x_1 d_x u,$$

$$u_2 = x_2 d_x u + x_1^2 \frac{d_x^2 u}{2\,!},$$

$$u_3 = x_3 d_x u + 2\,x_1 x_2 \frac{d_x^2 u}{2\,!} + x_1^3 \frac{d_x^3 u}{3\,!},$$

$$u_4 = x_4 d_x u + (2\,x_1 x_3 + x_2^2) \frac{d_x^2 u}{2\,!} + 3\,x_1^2 x_2 \frac{d_x^3 u}{3\,!} + x_1^4 \frac{d_x^4 u}{4\,!},$$

$$u_5 = x_5 d_x u + 2\,(x_1 x_4 + x_2 x_3) \frac{d_x^2 u}{2\,!} + 3\,(x_1^2 x_3 + x_1 x_2^2) \frac{d_x^3 u}{3\,!} + 4\,x_1^3 x_2 \frac{d_x^4 u}{4\,!} + x_1^5 \frac{d_x^5 u}{5\,!},$$

and generally

$$u_n = S'_n . d_x u + S^2_n . \frac{d^2_x u}{2!} + S^3_n . \frac{d^3_x u}{3!} + S^4_n . \frac{d^4_x u}{4!} + \ldots . + S^n_n . \frac{d^n_x u}{n!};$$

$$\therefore d_x u = \frac{u_1}{x_1}; \qquad \frac{d^2_x u}{2!} = \frac{\begin{vmatrix} x_1 & u_1 \\ x_2 & u_2 \end{vmatrix}}{x_1 . x_1^2};$$

$$\frac{d^3_x u}{3!} = \frac{\begin{vmatrix} x_1 & 0 & u_1 \\ x_2 & x_1^2 & u_2 \\ x_3 & 2\,x_1x_2 & u_3 \end{vmatrix}}{x_1 . x_1^2 . x_1^3}; \qquad \frac{d^4_x u}{4!} = \frac{\begin{vmatrix} x_1 & 0 & 0 & u_1 \\ x_2 & x_1^2 & 0 & u_2 \\ x_3 & 2\,x_1x_2 & a_1^3 & u_3 \\ x_4 & 2\,x_1x_3 + x_2^2 & 3\,x_1^2x_2 & u_4 \end{vmatrix}}{x_1 . x_1^2 . x_1^3 . x_1^4};$$

$$\frac{d^5_x u}{5!} = \frac{\begin{vmatrix} x_1 & 0 & 0 & 0 & u_1 \\ x_2 & x_1^2 & 0 & 0 & u_2 \\ x_3 & 2\,x_1x_2 & x_1^3 & 0 & u_3 \\ x_4 & 2\,x_1x_3 + x_2^2 & 3\,x_1^2\,x_2 & x_1^4 & u_4 \\ x_5 & 2\,(x_1x_4 + x_2x_3) & 3\,(x_1^2x_3 + x_1x_2^2) & 4\,x_1^3x_2 & u_5 \end{vmatrix}}{x_1 . x_1^2 . x_1^3 . x_1^4 . x_1^5};$$

and generally

$$\frac{d^n_x u}{n!} = \frac{\begin{vmatrix} S^1_1 & & & & u_1 \\ S^1_2 & S^2_2 & & & u_2 \\ S^1_3 & S^2_3 & S^3_3 & & u_3 \\ \cdot & \cdot & \cdot & \cdot\;\cdot\;\cdot\;\cdot & \cdot \\ S^1_{n-1} & S^2_{n-1} & S^3_{n-1} & S^{n-1}_{n-1} & u_{n-1} \\ S^1_n & S^2_n & S^3_n & S^{n-1}_n & u_n \end{vmatrix}}{x_1 . x_1^2 . x_1^3 \; . \; . \; . \; . \; x_1^n},$$

which may be written

$$\frac{d^n_x u}{n!} = \{ S^1_1 . S^2_2 . S^3_3 \ldots . S^{n-1}_{n-1} . u_n - S^1_1 . S^2_2 . S^3_3 \ldots S^{n-2}_{n-2} . u_{n-1} \,|\, S^{n-1}_n \,| + S^1_1 . S^2_2 . S^3_3 \ldots . S^{n-3}_{n-3} . u_{n-2} \,|\, S^{n-2}_{n-1}, S^{n-1}_n \,|$$
$$\ldots . (-)^{p-1} S^1_1 . S^2_2 . S^3_3 \ldots . S^{n-p}_{n-p} . u_{n-p+1} \,|\, S^{n-p+1}_{n-p+2}, S^{n-p+2}_{n-p+3}, \ldots . S^{n-1}_n \,| + \ldots . \} \div S^1_1 . S^2_2 . S^3_3 \ldots . S^n_n .$$

If $u = y$, this reduces to the last term, i. e., since in this case $u_1 = 1$, to

$$\frac{d^n_x u}{n!} = (-)^{n-1} \,|\, S^1_2, S^2_3, S^3_4, \ldots . \; S^{n-1}_n \,| \div S^1_1 . S^2_2 . S^3_3 \ldots . \; S^n_n .$$